

Quantum computation by quantum-like systems

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Abstract

Using a quantumlike description for light propagation in nonhomogeneous optical fibers, quantum information processing can be implemented by optical means. Quantum-like bits (qubits) are associated to light modes in the optical fiber and quantum gates to segments of the fiber providing an unitary transformation of the mode structure along a space direction. Simulation of nonlinear quantum effects is also discussed.

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1 Introduction

The quantum computer idea [1, 2, 3] uses the possibility to code numerical information by vectors in Hilbert space. In the simplest case, a two-dimensional Hilbert space is said to code a *qubit*. A physical realization of a qubit might be a spin 1/2 particle or a two-level atom. Calculations are carried out by unitary time-evolution transforming an input state vector into a final state vector in Hilbert space. Several characteristic features distinguish the quantum computation paradigm from classical computation. First, because of

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the superposition principle, the qubit space contains all the complex linear combinations, rather than just two states as in a classical bit. Second, in a quantum space of n qubits (of dimensionality 2^n) there are both factorized and entangled states, the latter having no correspondence in the space of n classical bits. Third, quantum evolution operates simultaneously in the 2^n states, implying intrinsic exponential parallelism of quantum computation. To extract the result from a quantum computation one has to observe the system, that is, to project it in one of the exponentially many states, thus losing most of the exponential amount of information generated by the unitary evolution. However, it is possible to take advantage of the exponentiality of quantum computation using the interference mechanism characteristic of quantum mechanics. In short, it is the combined effect of *entanglement*, *exponential parallelism* and *interference* that may allow the full potential of quantum computation to be realized [4].

According to the Church thesis, a classical Turing machine can simulate the computation of any computable function. Therefore a classical computer can simulate any computation of a quantum computer. The problem is how long the simulation will take to run. In particular it is known that some problems that are computed in exponential time by classical computers might be solved in polynomial time by quantum computers. Therefore, to take full advantage of the quantum computation algorithms, one requires physical devices obeying quantum laws.

There are many classical systems that physically implement some of the features of quantum computation[5, 6, 7, 8]. For example electromagnetic waves may be linearly superimposed and interfere. Nevertheless, given the qualitative differences between classical and quantum mechanics, no classical system, *where computations correspond to evolution in real time*, may ever implement simultaneously all the features of quantum computation. Otherwise we would have proved the physical equivalence of classical and quantum mechanics.

Notice however that quantum computation *is not* quantum mechanics. Quantum computation is a mathematical algorithm that uses all the mathematical features of quantum mechanics. In particular it is irrelevant for the algorithm how the Hilbert space is physically implemented and whether the unitary evolution is taking place along a real time direction or along some other coordinate. It is here that quantum-like systems may play a role. Quantum-like systems are classical systems which obey equations formally

identical to the Schrödinger equation, but where the role of time is played by a space coordinate. Therefore, insofar as they obey equations mathematically identical to those of quantum mechanics, they may implement all the quantum computation operations, provided the unitary evolution is interpreted not as evolution in time, but as evolution along a space coordinate. Because of this exchange of the role of the coordinates, there is no contradiction with the non-equivalence of classical and quantum mechanics.

In Sect.2 we give an overview of quantum-like systems and then, concentrating on fiber optics phenomena, we discuss how quantum information may be coded on the fiber and the kind of non-homogeneities and interactions needed to implement a set of universal quantum gates. Here we have concentrated on harmonic light modes. An alternative scheme might be developed based on soliton propagation on the fibers.

The fact that the unitary evolution needed for a quantum gate is obtained by setting up a nonuniform refraction index profile along the fiber, means that the unitary evolution becomes permanently coded in the hardware. This may be physically more convenient than to control a sequence of operations each time the gate is operated.

Abrams and Lloyd[9] have suggested that a stronger computation model would be obtained with non-linear terms in the quantum evolution. So far, all experimental evidence favors strict linearity of quantum mechanics. However, this is not so in quantum-like systems where non-linear effects may easily be introduced. Actual implementations of non-linear quantum computation might therefore use quantum-like systems.

2 Quantum-like systems

More than half a century ago, Fock and Leontovich have shown that paraxial beams of electromagnetic radiation in the parabolic approximation can be described by a Schrödinger-like equation [10, 11]. The role of time in this equation is played by the spatial (longitudinal) coordinate of the light beam, the role of Planck's constant is played by the light wavelength, and the role of potential energy by the index of refraction of the medium. Thus, the paraxial beam of light, a purely classical object, is a quantum-like system obeying equations formally identical to those of quantum mechanics.

Given the Helmholtz equation for a component of the electric field, ob-

tained for a fixed frequency, neglecting media dispersion and polarization

$$\frac{\partial^2 E}{\partial^2 x} + \frac{\partial^2 E}{\partial z^2} + k^2 n^2(x, z) E = 0 \quad (1)$$

(we use the planar configuration, $\lambda = 2\pi/k$ is the wavelength in vacuum, and z the longitudinal coordinate).

Introduce the complex function $\psi(x, z)$, which is the slowly varying amplitude of the electric field in

$$E(x, z) = n_0^{-1/2}(z) \psi(x, z) \exp \left[ik \int_0^z n_0(\xi) d\xi \right] \quad (2)$$

The ansatz (2) reduces the Helmholtz equation (1) to a Schrödinger-like equation

$$i\lambda \frac{\partial \psi(x, z)}{\partial z} = -\frac{\lambda^2}{4\pi n_0(z)} \frac{\partial^2 \psi(x, z)}{\partial x^2} + U(x, z) \psi(x, z) \quad (3)$$

$U(x, z)$ being an effective potential related to the index of refraction of the medium, $n(x, z)$

$$U(x, z) = \frac{\pi}{n_0(z)} \left[n_0^2(z) - n^2(x, z) \right]$$

and $n_0(z) = n(0, z)$ the index of refraction at the beam axis. In deriving this equation, second-order z -derivatives of ψ and derivatives of the function $n_0(z)$ were neglected. This is justified for slow variation of the index of refraction along the beam axis over distances of order of one wavelength

$$\frac{\lambda}{n_0^2(z)} \left| \frac{dn_0(z)}{dz} \right| \ll 1.$$

The Fock–Leontovich approximation is a basis for the description of light-beam propagation in optical fibers [12, 13, 14, 15] leading to

$$\begin{aligned} i\lambda \frac{\partial \psi(x, y, z)}{\partial z} = & -\frac{\lambda^2}{4\pi n_0(z)} \left(\frac{\partial^2 \psi(x, y, z)}{\partial x^2} + \frac{\partial^2 \psi(x, y, z)}{\partial y^2} \right) \\ & + \frac{\pi}{n_0(z)} \left[n_0^2(z) - n^2(x, y, z) \right] \psi(x, y, z) \end{aligned} \quad (4)$$

which is Schrödinger-like for a wave function depending on the transversal coordinates x and y . The longitudinal z coordinate plays the role of time

in the Schrödinger equation. The unitary z -evolution of the electromagnetic complex amplitude is described by the evolution operator $\hat{U}(z)$

$$\hat{U}(z, z_0)\psi(x, y, z_0) = \psi(x, y, z),$$

associated to the Hamiltonian

$$\hat{H}(z) = \left(\frac{\hat{p}_x^2}{2} + \frac{\hat{p}_y^2}{2} \right) \frac{1}{n_0(z)} + U(x, y, z). \quad (5)$$

with $\hat{p}_x = -i\lambda \frac{\partial}{\partial x}$, $\hat{p}_y = -i\lambda \frac{\partial}{\partial y}$ and a potential function

$$U(x, y, z) = \frac{\pi}{n_0(z)} \left[n_0^2(z) - n^2(x, y, z) \right].$$

Other quantum-like systems are reviewed in [16, 17]. An important example is sound-wave propagation in acoustic waveguides [18]. Acoustic waves in the paraxial approximation are well described by a Schrödinger-like equation. Charged-particle beams were also recently discussed as quantumlike systems [19, 20]. Light beams inside diode lasers have been treated as quantumlike systems as well, due to the waveguide structure of their active region [21, 22]. As remarked in [23] the variety of quantumlike classical systems provides a wide range of possibilities for perfect simulation of quantum computation operations by classical means.

3 Modes and gates in optical fibers

In an optical fiber, the light modes, being solutions of the Schrödinger-like equation (4), have all the properties of quantum-mechanical wave functions including the entanglement phenomenon.

Light modes in the fiber are used to code quantum-like bits (*qubits*). How many qubits may live in one optical fiber? In the simplest case, which is considered here, light modes with a fixed frequency are considered. But, in the same optical fiber, light of different frequencies may be used. The Helmholtz equation holds for each frequency, with an index of refraction profile that may be different for different frequencies. Hence, in the same optical fiber, one may store many qubits simply by exploiting the propagation of light beams

with different frequencies. Interaction of photons with different frequencies may provide useful computation effects.

The Fock–Leontovich approximation is obtained from the Helmholtz equation for the components of the electric or magnetic field. This equation is a scalar approximation which neglects the tensorial structure of the dielectric constant and the polarization. Also neglected are time and space dispersion, related to the nonlocal linear response of the medium to electromagnetic perturbations. Taking into account these effects would provide an even richer framework to accommodate qubits in the fiber.

We consider now the scalar fixed frequency situation described by Eq. (4). When the index of refraction profile has the form of an inverse well, the optical fiber traps discrete modes $\psi_{n_1 n_2}(x, y, z)$, n_1 and n_2 being integer labels.

With light modes on a fiber and the unitary z –evolution associated to Eq.(4) one may perform quantum computation over continuous variables in a way similar to the one proposed for time evolution in Ref.[24]. All the physical interactions needed for the construction of polynomial Hamiltonians are available by the choice of the appropriate refraction profile and by Kerr interactions. Also, as explained below, by restricting oneself to finite-dimensional subspaces of excitations, one may perform the same operations as in quantum computation with discrete variables.

There are several ways to code information by light modes in a fiber which may be useful for quantum information processing, in particular those associated to different choices of basis in self-focusing potentials. The important self-focusing case is associated to quadratic potentials of the form

$$U(x, y, z) = a(z)x^2 + b(z)y^2 + d(z)xy + e(z)x + f(z)y + l(z), \quad (6)$$

and in this case, an explicit solution may be obtained for the four z –dependent integrals of motion[15]

$$\begin{pmatrix} \hat{x}_0(z) \\ \hat{y}_0(z) \end{pmatrix} = \hat{U}(z, z_0) \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \hat{U}^{-1}(z, z_0), \quad \begin{pmatrix} \hat{p}_{x0}(z) \\ \hat{p}_{y0}(z) \end{pmatrix} = \hat{U}(z, z_0) \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} \hat{U}^{-1}(z, z_0) \quad (7)$$

Defining the boson integrals of motion

$$\begin{pmatrix} a(z) \\ a^\dagger(z) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{x}_0(z) + i\hat{p}_{x0}(z) \\ \hat{x}_0(z) - i\hat{p}_{x0}(z) \end{pmatrix}, \quad \begin{pmatrix} b(z) \\ b^\dagger(z) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{y}_0(z) + i\hat{p}_{y0}(z) \\ \hat{y}_0(z) - i\hat{p}_{y0}(z) \end{pmatrix} \quad (8)$$

several basis may be constructed. The discrete Fock-state modes are solutions to the eigenvalue equation

$$a^\dagger(z)a(z) \mid n_1, n_2, z\rangle = n_1 \mid n_1, n_2, z\rangle, \quad (9)$$

$$b^\dagger(z)b(z) \mid n_1, n_2, z\rangle = n_2 \mid n_1, n_2, z\rangle, \quad n_1, n_2 = 0, 1, 2, \dots \quad (10)$$

The Fock-state modes are obtained from the fundamental mode $\mid 0, 0, z\rangle$ by

$$\mid n_1, n_2, z\rangle = \frac{a_1^{\dagger n_1}(z)b^{\dagger n_2}(z)}{\sqrt{n_1!n_2!}} \mid 0, 0, z\rangle. \quad (11)$$

A spin-like description of the Fock-state modes is related to $SU(2)$ -subgroup of the Weyl-symplectic group in two dimensions. This is related to the Jordan-Schwinger map

$$J_+(z) = a^\dagger(z)b(z); \quad J_-(z) = b^\dagger(z)a(z); \quad J_3(z) = \frac{1}{2} (a^\dagger(z)a(z) - b^\dagger(z)b(z)) \quad (12)$$

Irreducible representation spaces for this subgroup are spanned by states $\{\mid n_1, n_2, z\rangle; n_1 + n_2 = N\}$ for each fixed N .

On the other hand, coherent modes in the optical fiber are labelled by two complex numbers α and β ,

$$\mid \alpha, \beta, z\rangle = \exp [\alpha a^\dagger(z) - \alpha^* a(z)] \exp [\beta b^\dagger(z) - \beta^* b(z)] \mid 0, 0, z\rangle, \quad (13)$$

Using this basis, the self-focusing fiber could be considered a Gaussian channel for numerical information. For arbitrary choices of the complex numbers α and β this is an overcomplete set. However choosing the numbers on the von Neumann lattice[26]

$$\alpha_{m_1, n_1} = \frac{1}{\sqrt{2}} (n_1 + i2\pi m_1); \quad \beta_{m_2, n_2} = \frac{1}{\sqrt{2}} (n_2 + i2\pi m_2) \quad (14)$$

and excluding two pairs (for example $m_1 = n_1 = m_2 = n_2 = 0$) one obtains a discrete *complete* set of coherent modes, which might provide a basis for the coding of a large amount of quantum-like information in the fiber.

We now analyze the question of what unitary transformations may be obtained by evolution of the light modes along the fiber. The simplest possibility is by a change of the index of refraction profile. From Eq.(4) we may

write a path integral representation for the evolution of the light mode along the fiber

$$\psi(x, y, z) = G(x, x_0, y, y_0, z) \psi(x_0, y_0, z) \quad (15)$$

with

$$G(x, x_0, y, y_0, z) = \int_{(x_0, y_0, 0)}^{(x, y, z)} d^2 \vec{\xi} \exp \left\{ \frac{i}{\lambda} \int_0^z d\tau \left[\pi n_0(z) \dot{\vec{\xi}}^2(z) - U(\vec{\xi}, z) \right] \right\} \quad (16)$$

$\vec{\xi}$ being a two-dimensional vector on the fiber sections and

$$U(\vec{\xi}, z) = \frac{\pi}{n_0(z)} \left[n_0^2(z) - n^2(\vec{\xi}, z) \right] \quad (17)$$

One sees from Eq.(16) that adjusting the index of refraction profile changes not only the potential but also the coefficient of the kinetic term. Let us consider a self-focusing quadratic potential and define, as before, a^\dagger, a and b^\dagger, b to be creation and annihilation operators for harmonic modes along the x and y -directions respectively. For simplicity we drop the z argument in the operators. Then, changing the index of refraction profile gives us direct access to the generators

$$a^\dagger a + b^\dagger b; a^\dagger + a; (a^\dagger + a)^2; b^\dagger + b; (b^\dagger + b)^2; (a^\dagger + a)(b^\dagger + b) \quad (18)$$

By a simple reasoning using the Baker-Campbell-Hausdorff formula one concludes that by a non-uniform change of the index of refraction one may obtain in Eq.(16) all the operations of the Weyl-symplectic group in two dimensions. This group is not compact. Therefore the unitary representations of the full group are not finite-dimensional. This might be explored for information manipulation schemes where an unbounded number of states is manipulated. However for operations on a finite number of qubits, it is the compact subgroups that are important. In particular a useful subgroup is the $SU(2)$ group described before in Eq.(12). Finite-dimensional irreducible spaces for this subgroup are

$$a^{\dagger n} b^{\dagger m} |0\rangle, \quad n + m = 2k \quad (19)$$

for $k = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ with dimension $2k + 1$. In this finite-dimensional spaces all unitary operations may be implemented on the fiber by changing the index of refraction profile.

To perform universal quantum computation it is necessary, at least, to have arbitrary unitary transformations on a single qubit and a CNOT operation on two qubits[27]. According to the discussion above the $|0\rangle, |1\rangle$ qubit states may be coded, for example, as follows

$$|1\rangle = a^\dagger|0\rangle, \quad |0\rangle = b^\dagger|0\rangle \quad (20)$$

This being a $k = \frac{1}{2}$ two-dimensional representation, all unitary transformations may be performed in this space by the $SU(2)$ subgroup.

For the CNOT operation we may code the $|1\rangle$ –state of the control bit as the application of two energy quanta along the x – direction and the $|0\rangle$ –state of the control bit as the application of two energy quanta along the y –direction. For the target bit we use the same coding as in (20). Therefore

$$|11\rangle = a^{\dagger 3}|0\rangle, |01\rangle = a^{\dagger 2}b^\dagger|0\rangle, |10\rangle = a^\dagger b^{\dagger 2}|0\rangle, |00\rangle = b^{\dagger 3}|0\rangle \quad (21)$$

the first label in $|\alpha\beta\rangle$ being the label of the target qubit and the second the label of the control qubit. The subspace spanned by (21) is a four-dimensional $SU(2)$ –irreducible subspace. Therefore all unitary transformations may be implemented in this subspace and, in particular, the CNOT operation.

We have therefore proved that, with this coding, the self-focusing fiber is capable of universal quantum computation. With higher order potentials many other possibilities would be available. For example, an interacting term of the form $\eta a^\dagger a b^\dagger b$ appears in the Kerr Hamiltonian. This allows to perform phase operations on a^\dagger modes gated by the b^\dagger excitations. For example by coding the target qubit as

$$|1\rangle = \frac{1}{\sqrt{2}}(1 - a^\dagger)|0\rangle, \quad |0\rangle = \frac{1}{\sqrt{2}}(1 + a^\dagger)|0\rangle$$

and the control qubit as

$$|1\rangle = b^\dagger|0\rangle, \quad |0\rangle = |0\rangle$$

the operator $\exp(i\pi a^\dagger a b^\dagger b)$ implements the CNOT gate.

The self-focusing fiber is a versatile medium to code quantum-like information and this is the reason we have emphasized the existence of several light mode basis. In Eq.(20) above, a qubit is coded using modes in the x and y directions. We might as well have used x –modes only and coded

the qulbit using the quadratures \hat{x} and \hat{p}_x . Then, light propagation with the symmetric self-focusing Hamiltonian corresponding to the operator

$$U = e^{i[(\hat{p}_x^2/2 + (\hat{x}^2/2)]\pi/4}$$

yields

$$\begin{pmatrix} \hat{p}_x \\ \hat{x} \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{p}_x + \hat{x} \\ \hat{p}_x - \hat{x} \end{pmatrix},$$

a Hadamard gate transformation. Observation of such transformation can be done measuring the position and direction of rays in the optical fiber for Gaussian wave packets.

4 Conclusions

1) Classical systems that are quantum-like, in the sense that their evolution along a space direction is described by a Schrödinger equation, possess a high potential for information processing including quantum computation. A promising system of this type consists of a light beam propagating along an optical fiber. It should be noticed however that similar possibilities exist with other systems, for example acoustic waves propagating along an acoustic waveguide. Practical implementation problems to be addressed are the choice of the optical fiber and the definition of a coding standard for the qulbits. The variety of materials used and a fairly well developed optical fiber technology give us hope that the model Hamiltonians needed for the operations of quantum computing may be physically implemented in this medium.

2) The fact that the unitary evolution of the quantum-like systems is associated to a space dimension, means that the unitary transformation is implemented in the hardware, rather than requiring a precise sequence of temporal operations. Also and this might be useful for mass production purposes, once a non-uniform refraction index profile is set up on the material, many different gates may be obtained simply by cutting fiber segments of different lengths.

3) Finally we should also point out the easy possibility to imitate nonlinear quantum mechanics by means of the nonlinear media response to the electromagnetic radiation. This might provide, as suggested in [9], even more powerful quantum computation algorithms.

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